Convergence of normals

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Theorem Let $X_n \sim N(\mu_n, \sigma_n^2)$. If $X_n \to X$ weakly, then $X \sim N(\mu, \sigma^2)$, and $\mu_n \to \mu$, $\sigma_n \to \sigma$.

Proof. Since $X_n \to X$ weakly, we have $e^{i\mu_n t - \frac{1}{2}\sigma_n^2 t^2}$ converge to $\phi(t)$ for all t, where ϕ is the ch.f. of X. Now we choose arbitrary $s > 0$, and we have $e^{i\mu_n s - \frac{1}{2}\sigma_n^2 s^2} \to \phi(s)$, which implies $e^{-\frac{1}{2}\sigma_n^2 s^2} \to |\phi(s)|$. Therefore, $|\phi(s)| = e^{-\frac{1}{2}\sigma^2 s^2}$, where $\sigma_n \to \sigma$.

Since $e^{-\frac{1}{2}\sigma_n^2 s^2} > 0$ for all $s \in \mathbb{R}$, $n \in \mathbb{N}$, $e^{i\mu_n t}$ converges for all $t \in \mathbb{R}$. If there exists two subsequences $\{\mu_{1,j}\}_j$, $\{\mu_{2,j}\}_j$ of $\{\mu_n\}_n$ so that $\{\mu_{1,j}\}_j \to a$ and $\{\mu_{2,j}\}_j \to b$, $a \neq b$, then $e^{i\mu_n t}$ converges to e^{iat} and e^{ibt} for every $t \in \mathbb{R}$, which is absurd. If $\mu_n \to \infty$ or $\mu_n \to -\infty$, then $P(X_n \in (c, d)) \to 0$ for every finite interval (c, d) (here we also use the fact we've proved: $\sigma_n \to \sigma$), and this contradicts the fact that X_n converges weakly to some X.

Therefore, the only possibility is that μ_n converges to some finite μ, and we have $e^{i\mu_n t - \frac{1}{2}\sigma_n^2 t^2} \to e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$ for all $t \in \mathbb{R}$. $\square$

Remark If we take $\mu_1 := 2\pi$, $\mu_n = l.c.m(2, 3, \cdots, n) \cdot (2\pi) + 1$, then we would find that $e^{i\mu_n t}$ converges to e^{it} for all $t \in \mathbb{Q}$. However, the convergence does not hold for every $t \in \mathbb{R}$, for otherwise this would be a contradiction to the above theorem.